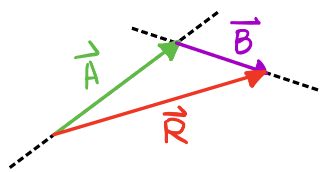
# VECTOR ADDITION

WE ARE GOING TO EXAMINE SEVERAL DIFFERENT METHODS FOR ADDING VECTORS. EACH OF THEM IS A TOOL FOR YOUR STATICS TOOLBOX, AND YOU SHOULD LEARN HOW AND WHEN TO USE EACH OF THEM.

ALL METHODS OF VECTOR ADDITION ARE ULTIMATELY BASED ON THE TIP-TO-TAIL TECHNIQUE PRESENTED IN THE SECTION ON 1-DIMENSIONAL VECTORS. THERE ARE TWO VISUAL METHODS TO ADD 2-D AND 3-D VECTORS:

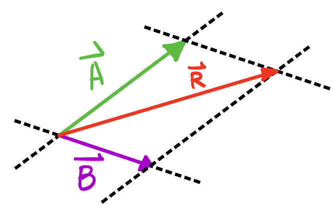
$$\vec{R} = \vec{A} + \vec{B}$$

## TRIANGLE RULE




PLACE THE TAIL OF $\vec{B}$ AT THE TIP OF $\vec{A}$. DRAW THE RESULTANT $\vec{R}$ FROM THE TAIL OF $\vec{A}$ TO THE TIP OF $\vec{B}$.

## PARALLELOGRAM RULE




PLACE BOTH VECTOR'S TAILS AT THE SAME POINT. COMPLETE THE OTHER TWO SIDES OF THE PARALLELOGRAM, AND THEN DRAW $\vec{R}$ FROM THE TAILS OF $\vec{A}$ AND $\vec{B}$ TO THE OPPOSITE CORNER OF THE PARALLELOGRAM.

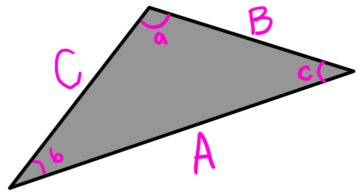
IF YOU CAREFULLY DRAW THE VECTORS TO SCALE WITH A RULER & PROTRACTOR, YOU CAN USE THE TRIANGLE METHOD OR THE PARALLELOGRAM METHOD TO GET A REASONABLY ACCURATE ESTIMATE OF $\vec{R}$ (IN 2-DIMENSIONS). CAD PROGRAMS CAN ALSO DO THIS FOR YOU.

ANOTHER (USUALLY QUICKER) METHOD IS TO USE TRIGONOMETRY BY:

1. DRAWING A QUICK DIAGRAM (TRIANGLE OR PARALLELOGRAM)
2. IDENTIFYING 3 KNOWN SIDES OR ANGLES
3. USING TRIG TO SOLVE FOR UNKNOWN SIDES OR ANGLES

# TRIG. REVIEW

FOR ANY OBLIQUE TRIANGLE, WE CAN USE THE LAW OF SINES AND THE LAW OF COSINES TO SOLVE FOR UNKNOWN SIDE LENGTHS & ANGLES.




THE LAW OF SINES STATES:

$$\frac{\sin a}{A} = \frac{\sin b}{B} = \frac{\sin c}{C}$$

THE LAW OF COSINES STATES:

$$A^2 = B^2 + C^2 - 2BC\cos a$$
$$B^2 = C^2 + A^2 - 2CA\cos b$$
$$C^2 = A^2 + B^2 - 2AB\cos c$$

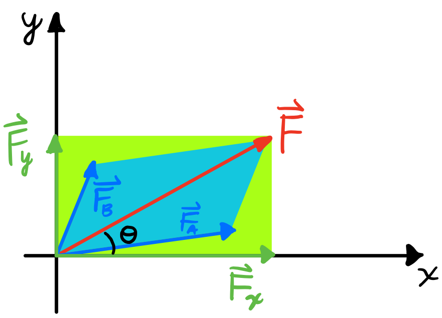
THE METHODS PRESENTED THUS FAR ARE POWERFUL, BUT LIMITED. TO NAME A FEW OF THE LIMITATIONS:

- WE CAN ONLY ADD TWO VECTORS AT A TIME. IF WE NEED TO ADD 3 OR MORE VECTORS, WE HAVE TO FIRST ADD TWO, THEN ADD A THIRD TO THE RESULTANT OF THE FIRST TWO, AND SO ON.
- IF WE MISLABEL OR MISIDENTIFY THE VECTORS, WE MAY GET AN INCORRECT ANSWER.
- THE LAW OF SINES AND THE LAW OF COSINES WORK WITH SCALARS. WE CAN USE THOSE SCALARS TO CONSTRUCT A VECTOR, BUT WE ARE NOT DIRECTLY WORKING WITH VECTORS.

THE ALGEBRAIC METHOD IS MUCH BETTER SUITED TO ADDING 3 OR MORE VECTORS. BEFORE WE CAN USE IT, HOWEVER, WE NEED TO LEARN ABOUT VECTOR RESOLUTION.

THE PROCESS OF FINDING COMPONENTS OF A VECTOR IN PARTICULAR DIRECTIONS IS CALLED **VECTOR RESOLUTION**. IT IS SIMILAR TO THE PARALLELOGRAM RULE, BUT REVERSED.

GIVEN A 2-D VECTOR $\vec{F}$, WE CAN RESOLVE IT INTO 2 COMPONENT VECTORS, WHICH ARE THE SIDES OF A PARALLELOGRAM WITH $\vec{F}$ AS THE DIAGONAL. WE CAN PICK ANY TWO DIRECTIONS FOR THE SIDES, BUT IT'S VERY USEFUL TO CHOOSE ORTHOGONAL (PERPENDICULAR) DIRECTIONS. THIS RESOLVES THE VECTOR $\vec{F}$ INTO **RECTANGULAR COMPONENTS**.



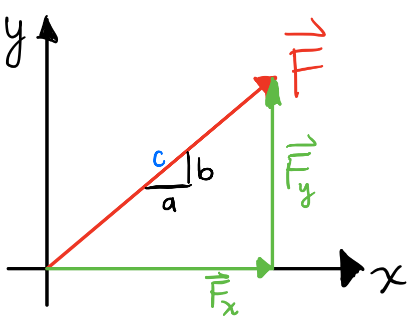

$$\vec{F} = \vec{F}_A + \vec{F}_B = \underbrace{\vec{F}_x + \vec{F}_y}_{\text{RECTANGULAR COMPONENTS}}$$

BECAUSE THE RECTANGULAR COMPONENTS FORM RIGHT TRIANGLES WITH $\vec{F}$, WE CAN FIND THEIR MAGNITUDES USING:

$$F_x = F\cos\theta \qquad F_y = F\sin\theta$$

INSTEAD OF AN ANGLE θ, YOU MAY BE GIVEN A SMALL "SLOPE" TRIANGLE. IN THIS CASE, YOU CAN USE THE RULE OF SIMILAR TRIANGLES TO FIND THE RECTANGULAR COMPONENTS OF A VECTOR

$$c = \sqrt{a^2 + b^2} \quad \text{(PYTHAGOREAN THEOREM)}$$




$$\frac{F_x}{F} = \frac{a}{c}$$

$$\frac{F_y}{F} = \frac{b}{c}$$

NOW, TO USE THE ALGEBRAIC METHOD TO ADD VECTORS TOGETHER, WE NEED TO DO 2 THINGS:

1. RESOLVE EACH VECTOR INTO ITS CARTESIAN COORDINATES

$$\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k} \qquad \vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$$

2. ALGEBRAICALLY SUM THE SCALAR COMPONENTS IN EACH COORDINATE DIRECTION.

$$\vec{R} = \vec{A} + \vec{B} = (A_x + B_x)\hat{\imath} + (A_y + B_y)\hat{\jmath} + (A_z + B_z)\hat{k}$$

AS WITH 1-DIMENSIONAL VECTORS, THE EASIEST WAY TO SUBTRACT A VECTOR FROM ANOTHER VECTOR IS TO MULTIPLY THE VECTOR YOU WANT TO SUBTRACT BY $-1$, AND THEN ADD THE TWO VECTORS TOGETHER USING ONE OF THE METHODS DESCRIBED ABOVE.

$$\vec{A} - \vec{B} = \vec{A} + (-1)\vec{B}$$

# DOT PRODUCT

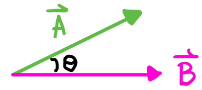
UNLIKE SCALAR ALGEBRA, WHERE THERE IS ONLY ONE WAY TO MULTIPLY NUMBERS TOGETHER, THERE ARE TWO WAYS WE CAN MULTIPLY VECTORS: THE DOT PRODUCT AND THE CROSS PRODUCT.

TO CALCULATE THE DOT PRODUCT OF TWO VECTORS $\vec{A} = A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k}$ AND $\vec{B} = B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k}$, SUM THE PRODUCTS OF THE COMPONENTS:

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

ALTERNATIVELY (AND EQUIVALENTLY), USE THE ANGLE $\theta$ BETWEEN THE TWO VECTORS TO CALCULATE THE DOT PRODUCT:

$$\vec{A} \cdot \vec{B} = |\vec{A}||\vec{B}|\cos\theta = AB\cos\theta$$




FROM THIS FORM OF THE DOT PRODUCT, WE CAN SEE THAT THE DOT PRODUCT OF TWO PERPENDICULAR VECTORS ($\theta = 90°$) IS ZERO ($\cos 90° = 0$).

WHEN DOTTING THE CARTESIAN UNIT VECTORS ($\hat{\imath}, \hat{\jmath}, \& \hat{k}$), WE HAVE THE FOLLOWING:

$$\begin{array}{lll} \hat{\imath}\cdot\hat{\imath} = 1 & \hat{\jmath}\cdot\hat{\imath} = 0 & \hat{k}\cdot\hat{\imath} = 0 \\ \hat{\imath}\cdot\hat{\jmath} = 0 & \hat{\jmath}\cdot\hat{\jmath} = 1 & \hat{k}\cdot\hat{\jmath} = 0 \\ \hat{\imath}\cdot\hat{k} = 0 & \hat{\jmath}\cdot\hat{k} = 0 & \hat{k}\cdot\hat{k} = 1 \end{array}$$

## WANT PROOF?

WE CAN USE REGULAR MULTIPLICATION TO PROVE OUR FIRST EQUATION FOR THE DOT PRODUCT:

$$\vec{A} \cdot \vec{B} = (A_x\hat{\imath} + A_y\hat{\jmath} + A_z\hat{k})(B_x\hat{\imath} + B_y\hat{\jmath} + B_z\hat{k})$$

$$= A_xB_x\,\hat{\imath}\cdot\hat{\imath} + A_xB_y\cancelto{0}{\hat{\imath}\cdot\hat{\jmath}} + A_xB_z\cancelto{0}{\hat{\imath}\cdot\hat{k}}$$
$$+ A_yB_x\cancelto{0}{\hat{\jmath}\cdot\hat{\imath}} + A_yB_y\,\hat{\jmath}\cdot\hat{\jmath} + A_yB_z\cancelto{0}{\hat{\jmath}\cdot\hat{k}}$$
$$+ A_zB_x\cancelto{0}{\hat{k}\cdot\hat{\imath}} + A_zB_y\cancelto{0}{\hat{k}\cdot\hat{\jmath}} + A_zB_z\,\hat{k}\cdot\hat{k}$$

$$= A_xB_x\underbrace{\hat{\imath}\cdot\hat{\imath}}_{=1} + A_yB_y\underbrace{\hat{\jmath}\cdot\hat{\jmath}}_{=1} + A_zB_z\underbrace{\hat{k}\cdot\hat{k}}_{=1}$$

$$\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z$$

DOT PRODUCTS ARE COMMUTATIVE:

$$\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$$

DOT PRODUCTS ARE ASSOCIATIVE:

$$c(\vec{A} \cdot \vec{B}) = (c\vec{A}) \cdot \vec{B} = \vec{A} \cdot (c\vec{B})$$

DOT PRODUCTS ARE DISTRIBUTIVE:

$$\vec{A} \cdot (\vec{B} + \vec{C}) = (\vec{A} \cdot \vec{B}) + (\vec{A} \cdot \vec{C})$$

NEXT, WE'LL LOOK AT WHY DOT PRODUCTS ARE OFTEN USED.

DOT PRODUCTS CAN BE USED TO FIND THE MAGNITUDE OF A VECTOR.

$$|\vec{A}| = \sqrt{\vec{A}\cdot\vec{A}}$$

WANT PROOF?

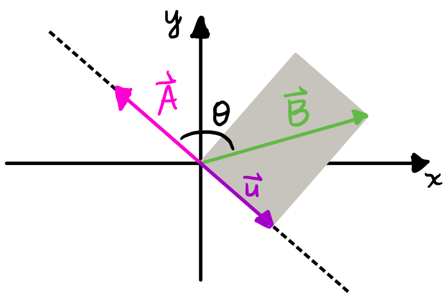
$$\vec{A}\cdot\vec{A} = A_x A_x + A_y A_y + A_z A_z$$
$$= A_x^2 + A_y^2 + A_z^2$$
$$\sqrt{\vec{A}\cdot\vec{A}} = \sqrt{A_x^2 + A_y^2 + A_z^2} = |\vec{A}|$$

DOT PRODUCTS CAN ALSO BE USED TO FIND THE ANGLE BETWEEN TWO VECTORS:

$$\vec{A}\cdot\vec{B} = |\vec{A}||\vec{B}|\cos\theta$$
$$\cos\theta = \frac{\vec{A}\cdot\vec{B}}{|\vec{A}||\vec{B}|}$$

THE DOT PRODUCT IS USED TO FIND THE PROJECTION OF ONE VECTOR ONTO ANOTHER.

LET'S LOOK AT AN EXAMPLE TO ILLUSTRATE THIS, WHERE $\vec{u}$ IS THE PROJECTION OF $\vec{B}$ ON $\vec{A}$




YOU CAN THINK OF $\vec{u}$ AS A VECTOR THE LENGTH OF THE SHADOW THAT $\vec{B}$ CASTS ON THE LINE OF ACTION OF $\vec{A}$.

MORE PRECISELY, IT IS THE RECTANGULAR COMPONENT OF $\vec{B}$ THAT IS PARALLEL TO $\vec{A}$.

THE LENGTH OF $\vec{u}$ IS SIMPLY:

$$|\vec{u}| = |\vec{B}|\cos\theta$$

THIS IS EQUIVALENT TO $\hat{A}\cdot\vec{B}$:

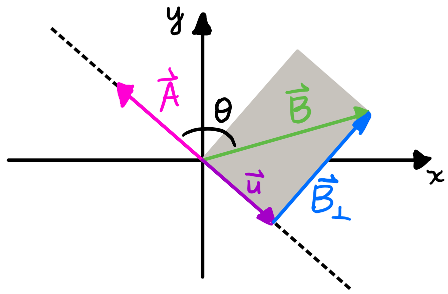
$$\hat{A}\cdot\vec{B} = \underbrace{|\hat{A}|}_{=1}|\vec{B}|\cos\theta = |\vec{B}|\cos\theta = |\vec{u}|$$

(MAGNITUDE OF ANY UNIT VECTOR = 1)

AND $\vec{u}$ HAS THE SAME DIRECTION AS $\vec{A}$, SO

$$\vec{u} = |\vec{u}|\hat{u} = |\vec{u}|\hat{A}$$
$$\vec{u} = (\hat{A}\cdot\vec{B})\hat{A}$$

FINALLY, WE CAN USE DOT PRODUCTS TO FIND THE COMPONENT OF A VECTOR THAT IS PERPENDICULAR TO ANOTHER. LOOKING AT OUR CURRENT EXAMPLE, $\vec{B}_{\perp}$ IS THE COMPONENT OF $\vec{B}$ THAT IS PERPENDICULAR TO $\vec{A}$.




MATHEMATICALLY,

$$\vec{B} = \vec{B}_{\perp} + \vec{u}$$

SOLVING FOR $\vec{B}_{\perp}$,

$$\vec{B}_{\perp} = \vec{B} - \vec{u}$$

$$= \vec{B} - (\hat{A} \cdot \vec{B})\hat{A}$$